Int. J. Solids Structures, 1973, Vol. 9, pp. 1403 to 1420. Pergamon Press. Printed in Great Britain

DISCRETIZED INITIAL-VALUE ANALYSIS OF CABLE NETS

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Abstract—A method of analysis for single-layer cable nets is presented based on a discrete mathematical model. The governing equations of equilibrium are nonlinear in the displacement terms and are solved by the Newton–Raphson method. Each linear cycle is a boundary-value problem which is solved as a set of initial-value problems. The method of analysis is primarily concerned with the determination of the forces and displacements that result from live loads and temperature changes; however, it can also be used to determine the initial dead load configuration. Example problems illustrate both applications.

1. INTRODUCTION

IN RECENT years, roofs supported by intersecting cables have become popular for covering large areas. This trend will undoubtedly continue because of the potential economic advantages and the esthetic qualities afforded by cable structures. A recent report [1] underscores the current interest that exists concerning suspended roof construction.

In the analysis of cable-supported structures, there are two different approaches that have gained popularity. One uses a mathematical model based on a discrete system, while the other employs a continuous system as a model. There are a variety of detailed methods under each category and the basic assumptions of the different investigators differ considerably. However, in most cases, the nonlinear response characteristics of the structure are considered.

Discrete formulations [2–5] have the advantage of providing a very realistic representation of most cable structures. However, this approach generally leads to the solution of a large number of simultaneous nonlinear algebraic equations. For large structures this can be an enormous computational task. Continuous formulations [5, 6] lead to much simpler computational algorithms; however, they generally do not represent the structure as realistically as the discrete form.

West and Robinson [5, 7] employed a continuous model in the study of suspension bridges. The Newton–Raphson method was used in which the nonlinear response was traced through a series of linear solutions. In each linear solution, the governing boundary-value problem was solved as a set of initial-value problems. In an extension of this work, West and Caramanico [8] used the same computational scheme with a discrete model. This approach retains the realism of the discrete model without the disadvantage of having to solve simultaneous equations. The purpose of this paper is to extend the initial-value analysis to treat the cable net problem.

There have been other attempts to use a marching process in the solution of problems of this type. Some of these have failed numerically because of the growth of the solution. This problem, however, can be handled rather easily in each linear solution by a suppression technique.

Avent [9] used a "walk-through" procedure somewhat similar to the initial-value method. He represented the equations of equilibrium in finite difference form and considered only vertical displacements. In the work presented here, the equations of equilibrium are derived from energy considerations and *all* nonlinear terms are retained. Also, the full spatial response of the cable is considered.

The method presented is designed to determine the response of a cable net to applied loads and temperature changes. It can also be used to determine the initial dead load configuration of a cable net. Example problems are presented to illustrate typical applications.

2. DESCRIPTION OF MATHEMATICAL MODEL

The following items describe the structural model adopted for this study:

1. The primary cable system is composed of two sets of intersecting cables connected at the points of intersection to form a net-type structure. These cables are anchored to some support mechanism at their ends. Hangers, if employed, are attached to the net at the intersections of the main cables.
2. The entire cable system is composed of discrete structural elements joined by frictionless pins at the connection points. Each element is assumed to be loaded axially along its centroidal axis with the moment and shearing capacities neglected. The centroidal axes of all members framing into a joint are assumed to intersect at a point.
3. There is no relative movement of the intersecting elements at intersection points. However, these points are free to displace in any direction.
4. Loads can be applied in any direction at the points of intersection of the cable net system, at the hanger bases or at the supports. This includes the dead load of the system which is lumped at these points. Temperature changes, which are reflected as equivalent loads at the joints, are permitted in any member.
5. The support systems for the cables are idealized as linear springs.
6. Physical nonlinearity is ignored, with all materials assumed to obey Hooke's law. However, the geometric nonlinearity is fully treated.
7. The initial equilibrium configuration under the action of the initial prestressing forces in the cables and the dead load is determined if it is not already known. The live load and/or temperature change is applied to this configuration.

3. EQUATIONS OF EQUILIBRIUM

A typical joint (ij) is shown in Fig. 1 in which the ith transverse cable intersects the jth longitudinal cable. Node points along the ith transverse cable and the jth longitudinal cable are identified as (ir) and (rj), respectively, where r ranges over the number of cable

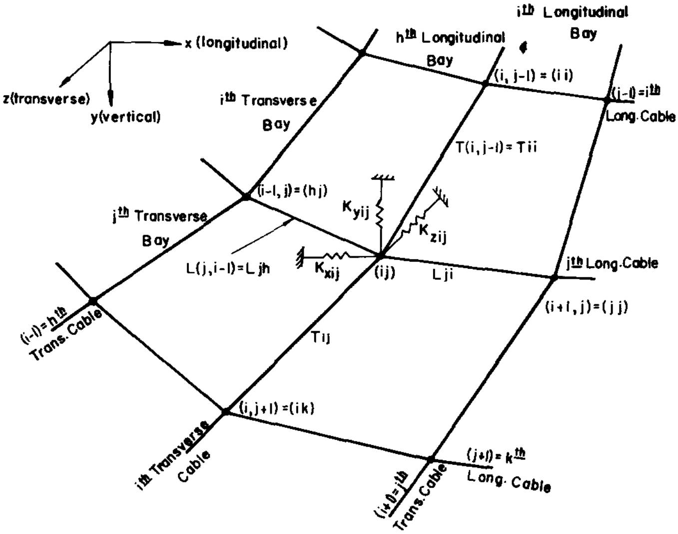

FIG. 1. Typical joint (ij).

intersections present on each cable. Similarly, segments of the ith transverse cable and segments of the jth longitudinal cable are identified by Tir and Ljr, respectively, where r ranges over the number of bays for each cable. Since in general, joint (ij) could be a point of external support, springs are included which represent the support mechanisms. These have stiffnesses of K_{xij}, K_{yij} and K_{zij} in the longitudinal, vertical and transverse directions, respectively. The effect of including hangers in the analysis is discussed later.

The longitudinal, vertical and transverse displacements that result from live load and temperature change are denoted by u, v and w, respectively. The applied live loads in the corresponding three directions are Q_x, Q_y and Q_z. These displacements and loads are subscripted to identify the corresponding joint as shown in Fig. 2 for joint (ij). Loads and displacements are positive in the sense shown in Fig. 2, and are both with respect to some known initial equilibrium configuration. This restriction will be relaxed later.

The detailed derivation of the equations of equilibrium for joint (ij) is outlined in the Appendix. These equations are given in an abbreviated matrix form as follows:

$$[C]\{\delta\}+\{N\} = \{T\} \tag{1}$$

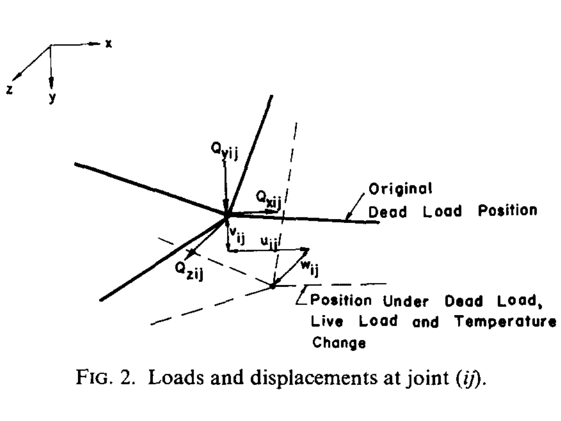

FIG. 2. Loads and displacements at joint (ij).

where

$$[C] = \begin{bmatrix} [C]_{11} & [C]_{12} & [C]_{13} \\ [C]_{21} & [C]_{22} & [C]_{23} \\ [C]_{31} & [C]_{32} & [C]_{33} \end{bmatrix}$$

$$[C]_{11} = [-\alpha_{Ljh} \mid -\alpha_{Tii} \mid (\alpha_{Ljh}+\alpha_{Lji}+\alpha_{Tii}+\alpha_{Tij}+K_{xij}) \mid -\alpha_{Tij} \mid -\alpha_{Lji}]$$

$$[C]_{22} = [-\gamma_{Ljh} \mid -\gamma_{Tii} \mid (\gamma_{Ljh}+\gamma_{Lji}+\gamma_{Tii}+\gamma_{Tij}+K_{yij}) \mid -\gamma_{Tij} \mid -\gamma_{Lji}]$$

$$[C]_{33} = [-\varepsilon_{Ljh} \mid -\varepsilon_{Tii} \mid (\varepsilon_{Ljh}+\varepsilon_{Lji}+\varepsilon_{Tii}+\varepsilon_{Tij}+K_{zij}) \mid -\varepsilon_{Tij} \mid -\varepsilon_{Lji}]$$

$$[C]_{12} = [-\beta_{Ljh} \mid -\beta_{Tii} \mid (\beta_{Ljh}+\beta_{Lji}+\beta_{Tii}+\beta_{Tij}) \mid -\beta_{Tij} \mid -\beta_{Lji}] = [C]_{21}$$

$$[C]_{13} = [-\xi_{Ljh} \mid -\xi_{Tii} \mid (\xi_{Ljh}+\xi_{Lji}+\xi_{Tii}+\xi_{Tij}) \mid -\xi_{Tij} \mid -\xi_{Lji}] = [C]_{31}$$

$$[C]_{23} = [-\Delta_{Ljh} \mid -\Delta_{Tii} \mid (\Delta_{Ljh}+\Delta_{Lji}+\Delta_{Tii}+\Delta_{Tij}) \mid -\Delta_{Tij} \mid -\Delta_{Lji}] = [C]_{32}$$

$$\{\delta\}^T = \{u_{hj}u_{ii}u_{ij}u_{ik}u_{jj}v_{hj}v_{ii}v_{ij}v_{ik}v_{jj}w_{hj}w_{ii}w_{ij}w_{ik}w_{jj}\}$$

$$\{N\}^T = \{N_{xij}N_{yij}N_{zij}\}$$

$$\{T\}^T = \{T_{xij}T_{yij}T_{zij}\}.$$

The $[C]$ matrix is composed of the terms α, β, γ, ε, Δ and ξ. These coefficients represent the stiffness of the cable segments framing into joint (ij) and are given in detail in the Appendix. They are triple subscripted, with the first subscript being L or T to indicate whether the cable segment is longitudinal or transverse. The remaining two subscripts specify the cable and bay as shown in Fig. 1.

The vectors $\{N\}$ and $\{T\}$ each have three elements with a triple subscript. The first subscript is either x, y or z to indicate the longitudinal, vertical or transverse directions, and the remaining subscript specifies the joint under consideration.

The method used to solve equation (1) requires that the equation be linearized by setting the nonlinear terms equal to zero. This results in the following simplified equation:

$$[C]\{\delta\} = \{T\}. \tag{2}$$

4. SOLUTION OF EQUATIONS

Equation (1) must be satisfied for all cable intersection points and support points throughout the grid system. In general, the loading is known, and the displacements are to be determined. Because of the nonlinear character of equation (1), the displacements cannot be determined by a direct solution. Instead, some iterative scheme must be employed. In this presentation, the Newton–Raphson method is used.

Newton–Raphson method

This procedure is a well-known technique for solving nonlinear equations. It is described in detail by West and Robinson [5, 7], and West and Caramanico [8] as it applies to cable-type problems. Only a brief discussion of the important features of the method is given here.

Equation (2) is a linearized form of equation (1) in which the vector of nonlinear terms, $\{N\}$, has been set equal to zero. A set of displacements, $\{\delta\}$, which satisfies this linearized equation at all points for some given set of loads, $\{T\}$, is a linear solution. A series of linear solutions of this type is carried out, with the loads for each linearization being the difference between the known loads applied to the initial equilibrium configuration and the loads corresponding to some deformed configuration. The $[C]$ matrix for each linear solution is evaluated at the deformed position about which the linearization is being performed.

The loads corresponding to any deformed configuration are determined from equation (1) by using the stiffness quantities corresponding to the initial equilibrium configuration and the total displacements corresponding to the deformed configuration.

After several linearizations, the loads for the next linearization are very small as compared to the original loading. At this point, the deformed structure can support the known applied loads on the system, and the accumulated displacements from all linear solutions constitute the solution to the full nonlinear equations.

In the above discussion, it has been implied that the full load is applied to the structure. In practice, it may often be better to segment the load. Each segment of load is treated as described above, with the initial configuration for the ith load segment corresponding to the final configuration for the $(i-1)$th load segment.

Linear solution

As explained earlier, the Newton–Raphson method requires the repeated determination of the displacements corresponding to a set of loads by solving the linear equations. Each linear solution is basically a boundary-value problem; however, it is solved as a set of initial-value problems.

The initial-value technique requires that equation (2) be solved in two different ways. These are described in the following:

a. *Equations for forward solution.* Figure 1 shows the arrangement of members framing into point (ij). This solution form is designed to compute the displacements at point (jj) from equation (2), which represents equilibrium at point (ij). It is assumed that all of the displacements in equation (2) are known except those at point (jj). With all known quantities shifted to the right-hand side, equation (2) reduces to 3 simultaneous algebraic equations with the three displacements at point (jj) as the unknowns. Solving these equations, we obtain

$$\begin{Bmatrix} u_{jj} \\ v_{jj} \\ w_{jj} \end{Bmatrix} = \begin{Bmatrix} u_{ij} \\ v_{ij} \\ w_{ij} \end{Bmatrix} + \frac{1}{\begin{pmatrix} -\alpha\gamma\varepsilon - 2\xi\beta\Delta \\ +\xi^2\gamma + \beta^2\varepsilon + \alpha\Delta^2 \end{pmatrix}_{Lji}} \left[\begin{array}{c|c|c} (\gamma\varepsilon - \Delta^2) & (\xi\Delta - \beta\varepsilon) & (\beta\Delta - \xi\gamma) \\ \hline (\xi\Delta - \beta\varepsilon) & (\alpha\varepsilon - \xi^2) & (\beta\xi - \alpha\Delta) \\ \hline (\beta\Delta - \xi\gamma) & (\beta\xi - \alpha\Delta) & (\alpha\gamma - \beta^2) \end{array}\right]_{Lji}$$

$$\times \begin{Bmatrix} \sum\limits_m \alpha_m(\bar{u}_m - u_{ij}) + \sum\limits_m \beta_m(\bar{v}_m - v_{ij}) + \sum\limits_m \xi_m(\bar{w}_m - w_{ij}) - K_{xij}u_{ij} + T_{xij} \\ \sum\limits_m \beta_m(\bar{u}_m - u_{ij}) + \sum\limits_m \gamma_m(\bar{v}_m - v_{ij}) + \sum\limits_m \Delta_m(\bar{w}_m - w_{ij}) - K_{yij}v_{ij} + T_{yij} \\ \sum\limits_m \xi_m(\bar{u}_m - u_{ij}) \quad \sum\limits_m \Delta_m(\bar{v}_m - v_{ij}) + \sum\limits_m \varepsilon_m(\bar{w}_m - w_{ij}) - K_{zij}w_{ij} + T_{zij} \end{Bmatrix}. \quad (3)$$

The subscript Lji on the (3×3) matrix and on the denominator which premultiplies this matrix indicates that all quantities within these units refer to member Lji. In the summation quantities, m ranges over members Ljh, Tii and Tij and $\bar{u}_m$, $\bar{v}_m$ and $\bar{w}_m$ are the longitudinal, vertical and transverse displacements, respectively, at the ends opposite from point (ij) on these members.

Equation (3) simply states that the displacements at point (jj) are equal to the displacements at point (ij) plus the effects of change in length of segment Lji. The (3×3) matrix and its scalar pre-multiplier reflect the flexibility of member Lji, indicating the displacements at point (jj) relative to point (ij) that result from unit loads at point (ij). The vector which post-multiplies the flexibility matrix contains the loads that are input at point (ij). The T terms are the applied loads for this linearization (Q and Q' terms of equations 8, 9 and 10) while all other terms represent loads that are generated at point (ij) from the deformation of the elements (except Lji) that frame into this point.

b. *Equations for central solution.* Attention is again focused on Fig. 1. This solution form is for computing the displacements at point (ij), the point for which equilibrium is expressed in equation (2). Here, it is assumed that all of the displacements in equation (2) are known except those at point (ij). Equation (2) thus reduces to 3 algebraic equations with the three displacements at point (ij) as the unknowns. Solving these.equations, we obtain

$$\begin{Bmatrix} u_{ij} \\ v_{ij} \\ w_{ij} \end{Bmatrix} = \frac{1}{\left(\begin{array}{c} AGF+2BCD \\ -C^2G-B^2F-D^2A \end{array}\right)} \left[\begin{array}{c|c|c} (GF-D^2) & (DC-BF) & (BD-GC) \\ \hline (DC-BF) & (AF-C^2) & (BC-AD) \\ \hline (BD-GC) & (BC-AD) & (AG-B^2) \end{array}\right]$$

$$\times \begin{Bmatrix} \sum_m \alpha_m \bar{u}_m + \sum_m \beta_m \bar{v}_m + \sum_m \xi_m \bar{w}_m + T_{xij} \\ \sum_m \beta_m \bar{u}_m + \sum_m \gamma_m \bar{v}_m + \sum_m \Delta_m \bar{w}_m + T_{yij} \\ \sum_m \xi_m \bar{u}_m + \sum_m \Delta_m \bar{v}_m + \sum_m \varepsilon_m \bar{w}_m + T_{zij} \end{Bmatrix} \tag{4}$$

where

$$\begin{aligned} A &= \alpha_{Ljh}+\alpha_{Lji}+\alpha_{Tii}+\alpha_{Tij}+K_{xij} \\ B &= \beta_{Ljh}+\beta_{Lji}+\beta_{Tii}+\beta_{Tij} \\ C &= \xi_{Ljh}+\xi_{Lji}+\xi_{Tii}+\xi_{Tij} \\ D &= \Delta_{Ljh}+\Delta_{Lji}+\Delta_{Tii}+\Delta_{Tij} \\ G &= \gamma_{Ljh}+\gamma_{Lji}+\gamma_{Tii}+\gamma_{Tij}+K_{yij} \\ F &= \varepsilon_{Ljh}+\varepsilon_{Lji}+\varepsilon_{Tii}+\varepsilon_{Tij}+K_{zij}. \end{aligned}$$

In the summation terms, m ranges over bars Ljh, Lji, Tii and Tij. The displacements $\bar{u}_m$, $\bar{v}_m$ and $\bar{w}_m$ are the far-end displacements with respect to point (ij) of these bars.

In equation (4), the (3×3) matrix and its pre-multiplier give the flexibility of joint (ij), reflecting the displacements at point (ij) that result from unit loads at this same point. The load vector which post-multiplies the flexibility matrix contains all the loads input at point (ij) either by direct application or by element deformations.

Initial-value approach

The manner in which the initial-value procedure is applied will be explained by use of an example problem. Consider the hyperbolic paraboloid net of Fig. 3. The longitudinal and transverse directions are selected as shown. Initial displacements are assumed at the left end of each longitudinal cable. These points are identified by the letter a in Fig. 3. If there are m longitudinal cables, a total of $3m$ displacement quantities are assumed. Equation (3) is applied at each of these points to determine the displacements at each interior point along the transverse cable 1–1. The displacements at the side points of transverse cable 1–1 can then be determined by applying equation (4) to the two points marked b along this cable. Now, by applying equation (3) at each intersection point along transverse cable 1–1, we can determine the displacements at the interior points along transverse cable 2–2. Application of equation (4) at the side points b of this cable then establishes the displacements at these points. This procedure continues until all of the displacements have been determined along transverse cable n–n. Equation (3) can now be applied at each interior point along transverse cable n–n to determine the displacements at the terminal points identified by the letter c.

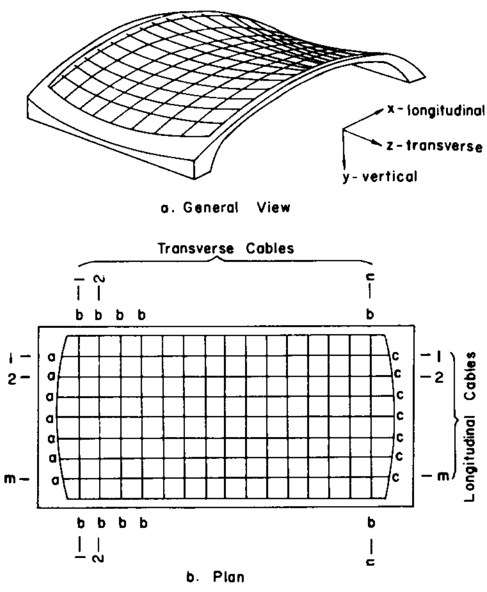

FIG. 3. Hyperbolic paraboloid net.

The displacements at the terminal points will satisfy the boundary conditions only if the correct initial values were assumed at points a. This is, of course, very unlikely. Thus, some procedure to modify the solution is essential. The technique used is as follows:

The solution described above uses the loads for the linearization being treated in all applications of equations (3 and 4). This shall be referred to as the particular solution. Simultaneously with the determination of the particular solution, $3m$ homogeneous solutions are generated. Each of these solutions corresponds to a unit value of one of the $3m$ initial displacements while all others are set equal to zero. Thus, each homogeneous solution will give the terminal displacements at points c that are induced by a unit value for one of the initial displacements.

The boundary conditions to be satisfied at the terminal points are force conditions. The particular solution and the $3m$ homogeneous solutions must be combined to yield the correct loads at the terminal points. Thus, we have

$$\sum_{i=1}^{3m} C_i\{P\}_i + \{P\}_o = \{\tilde{T}\} \tag{5}$$

where $\{P\}_i$ represents the forces induced at the terminal joints by the displacements corresponding to the ith homogeneous solution, $\{P\}_o$ gives the corresponding forces for the particular solution, $\{\tilde{T}\}$ is the applied loads for this linearization at all the terminal points, and C_i is a constant associated with the ith homogeneous solution. Equation (5) thus represents $3m$ simultaneous equations with C_i $(i = 1, 2 \ldots 3m)$ as the unknowns. Solving for C_i, we can combine the $(3m+1)$ linear solutions to obtain a final linear solution which will satisfy the terminal boundary conditions. Thus, we have

$$S = S_o + \sum_{i=1}^{3m} C_i S_i \tag{6}$$

where S is the final linear solution, S_o is the particular solution and S_i is the ith homogeneous solution.

In equation (5), the force vector $\{P\}_i$ is determined by applying equation (2) at each terminal point using the displacements of the ith homogeneous solution. The collection of the resulting $\{T\}$ vectors at all terminal points comprises $\{P\}_i$. The vector $\{P\}_o$ is similarly determined by using the displacements of the particular solution.

At the end of a given linearization, the algebraic sums of the displacements for all linear solutions are determined. These displacements are substituted into equation (1) with all terms being evaluated in terms of the original equilibrium configuration for this load segment. The resulting $\{T\}$ vectors for all joints give the loads corresponding to this deformed configuration. These loads are compared with the applied loads for this segment to determine whether another linearization is necessary.

Numerical difficulties

As previously discussed in detail by West and Robinson [5, 7] and West and Caramanico [8], the initial-value scheme has mathematical sensitivities. There are two ways in which numerical problems may arise.

First, there is a problem in selecting the displacements at the beginning points. This is clear from an examination of equation (3). Assume that point (ij) is a beginning point for which initial displacements are assumed. These displacements are multiplied by the very large support stiffnesses. If the assumed displacements are too large, the resulting forces will dominate the load vector. The effect is to make the applied T loads relatively insignificant.

The second problem occurs when the displacements become too large as the solution advances across the structure. Again, an examination of equation (3) explains the problem. When large displacement quantities are multiplied by the cable stiffness terms, the resulting loads can render the load vector insensitive to the T loads.

In either case, it is clear that displacements must be kept small enough that the effect of the T loads are not lost in round-off of larger load terms. When this occurs, the solution no longer reflects the actual applied loading on the structure.

The problem of the growth of displacements as the solution advances can be controlled quite easily by a "suppression" scheme. This has previously been treated in detail [5, 7, 8].

Additional topics

For the sake of simplicity, the discussion thus far has treated cable nets with flexible supports, with no hangers, and for which the initial equilibrium configuration is known. These restrictions will now be relaxed.

a. *Initial equilibrium configuration.* As explained in the Appendix, if the initial equilibrium configuration is known, the right-hand sides of equations (8, 9 and 10) are simplified such that only the applied loads (Q terms) and temperature loads (Q^t terms) remain. In the discussion thus far, this has been assumed to be the situation.

If the initial equilibrium configuration is not known, one proceeds in a slightly different fashion. Any approximate scheme is used to select an assumed initial shape with corresponding element forces. If the Q and Q^t forces of equations (8, 9 and 10) are set equal to zero, the right-hand sides of these equations reduce to residual loads, which measure the amount by which the assumed configuration is not in equilibrium under the initial loads, W. Using these residual loads as the T loads of equation (1), we can solve for a set of displacements which, when applied to the assumed configuration, will yield the correct initial equilibrium configuration. This is now the initial equilibrium configuration to which the Q and Q^t forces must be applied in the fashion described earlier.

If the initial configuration is not desired, then the complete right-hand sides of equations (8, 9 and 10) are taken as the T loads of equation (1). The resulting displacements give the final position of the structure under the initial dead load, live load and temperature change. These displacements are, of course, with respect to the assumed configuration.

b. *Hangers.* If a hanger is present at point (ij), three equations of equilibrium are available at the base of the hanger. For a linear solution, these equations will be similar to equation (4), with the K terms reflecting the stiffnesses of any member framing in at the hanger base. That is, the displacements are determined at the point for which equilibrium is expressed. Having established these hanger-base displacements, we use equation (3) to determine the displacements at point (jj). However, in this case the range of m in equation (3) must include the hanger element framing into point (ij).

c. *Rigid supports.* The case of rigid supports is actually simpler than the case of flexible supports. The procedure will be explained with the aid of Fig. 3. Here, all the displacements at point a are zero. Thus, the initially assumed displacements are at the interior points of transverse cable 1–1. Since points b are not free to displace, these points are never occupied and equation (4) is not used. Using equation (3) we advance across the structure as before. The boundary conditions at c are now displacement boundary conditions. The homogeneous and particular solutions are combined at these points to produce zero terminal displacements.

5. EXAMPLE PROBLEMS

In this section, the results of several numerical problems are presented. The method has been applied successfully to larger problems, but the intent here is only to illustrate

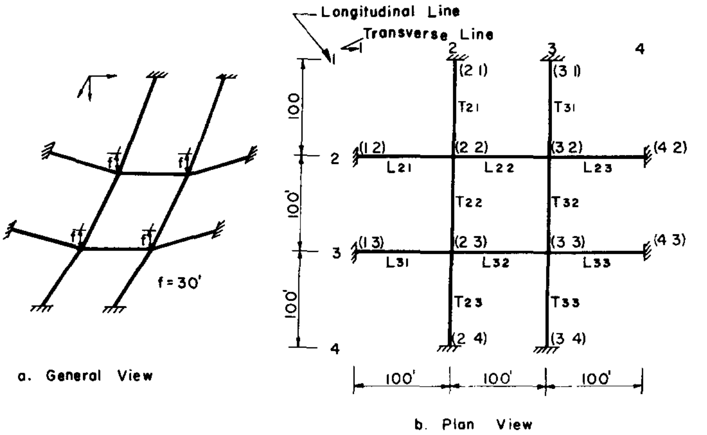

FIG. 4. Cable net for example problems 1 and 2.

how the method of analysis works and, in one instance, to compare the results obtained with those of another investigator.

Problem 1. The net shown in Fig. 4 has been used previously by Saafan [2] as an example of a finite deflection theory analysis. The numbering of joints and cable elements follows the general scheme outlined earlier. It is seen that longitudinal and transverse lines are established through the support points even though there are no cables along these lines. This is essential for cases where the supports are free to displace. The dimensions given in Fig. 4 and the additional data of Table 1 describe the assumed configuration employed by Saafan and also used by the authors. This assumed configuration is not in equilibrium under the prescribed loads and cable forces. Thus, residual loads are determined (right-hand sides of equations (8, 9 and 10) with Q and Q' terms equal to zero), and applied as a single load segment to determine the dead load configuration. The displacements from the assumed configuration to the dead load configuration along with the resulting cable forces are shown in Table 2. The final solution required two linearizations. Because the structure is symmetrical, only partial results are given. Also given in Table 2 are the results reported by Saafan [5]. The results of the two methods agree very well.

Problem 2. The results of Problem 1 establish the dead load configuration and corresponding cable forces. This arrangement becomes the initial equilibrium configuration to which live load is applied.

In this problem, the live load consists of a single vertical downward load of 12 kips at point (3 2).

The displacements, with respect to the dead load configuration, and the final cable forces are given in Tables 3 and 4. Since the loading is symmetrical with respect to a diagonal line through points (23) and (32), only partial results are given. These results required four linearizations within a single load segment.

Problem 3. The problems presented thus far were intended to illustrate the application of the method of analysis to a very elementary structure and to compare some of the results with those of another investigation. The structure used did not represent a typical cable roof net.

TABLE 1. DATA FOR ASSUMED CONFIGURATION OF PROBLEM 1

Description		Magnitude
Cross-sectional area of cables		0·227 in^2
Support stiffness		Rigid
Young's modulus of elasticity		12,000 $kips/in^2$
Prestressing force	Horizontal members	5·459 kips
	Inclined members	5·325 kips
Load acting vertically downward at joints (2 2), (2 3), (3 2) and (3 3)		8·0 kips

TABLE 2. JOINT DISPLACEMENTS AND CABLE FORCES AT DEAD LOAD FOR PROBLEM 1

Investigator	Displacements of joint (2 2) (ft)			Final cable forces in kips	
	Long.	Vert.	Trans.	*L*21	*L*22
West and Kar	−0·1325	1·4698	−0·1324	13·311	12·677
Saafan		1·4707		13·309	12·677

In this example problem, a saddle shaped net that is more representative of a real structure will be used. The net used is shown in Fig. 5 and covers an area of 150 ft square. Customarily, such a net would have 15–20 cables in each direction, however, in this example only 5 cables will be taken in each direction. This is done here for the sake of brevity, however, Natarajan [12] and Krishna [13] have shown that cable net structures can be

TABLE 3. JOINT DISPLACEMENTS FOR PROBLEM 2

Joint	Displacements (ft)		
	Long.	Vert.	Trans.
22	0·961	−1·827	0·468
23	−0·364	0·956	0·364
32	1·111	4·620	−1·111

TABLE 4. FINAL CABLE FORCES FOR PROBLEM 2

Cable element	Cable forces in kips
*L*21	23·439
*L*22	22·772
*L*31	11·915
*T*21	11·150
*T*22	11·130
*T*31	24·888

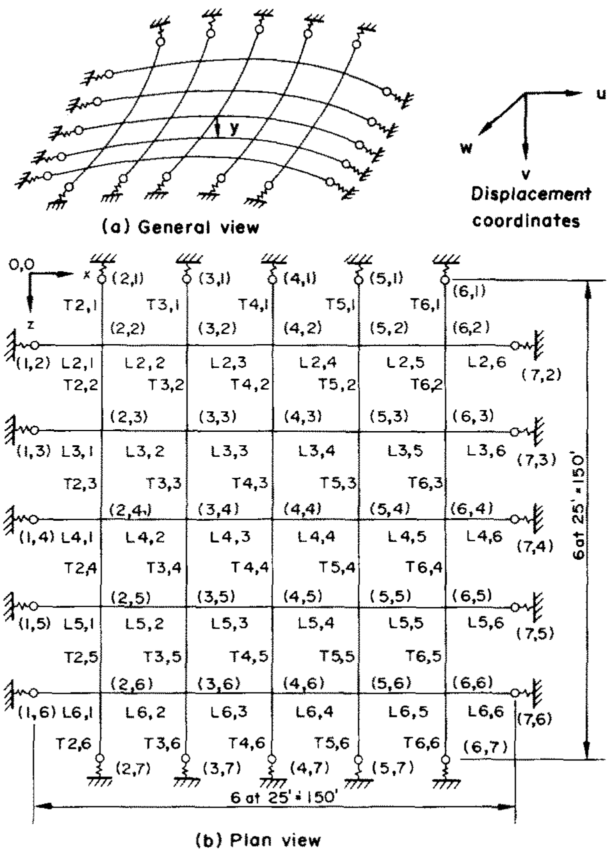

FIG. 5. Cable net for example problem 3.

satisfactorily represented by a net of fewer than the actual number of cables. This same point was shown to be true by West and Caramanico [8] in their studies on suspension bridges. However, their study indicated that some care must be exercised in the boundary regions of the structure.

The detailed information concerning the assumed initial configuration is given in Tables 5 and 6. Since this assumed configuration is not in equilibrium under the specified dead load given in Table 5, the correct equilibrium configuration must first be determined. The residual loads, which measure the amount by which the assumed configuration is not in equilibrium under the given dead loads, are determined from the right-hand sides of equations (8, 9 and 10) with the Q and Q' forces set equal to zero. These residual loads are then applied to the structure in its initially assumed state and the resulting displacements and member forces are determined. From these results, the adjusted joint coordinates and cable tensions corresponding to the dead load equilibrium state are computed. This data is summarized in Tables 7 and 8. Because of symmetry, the results are given for only one quadrant of the structure.

In this case, the residual loads were applied in a single load segment and three linearizations were necessary to arrive at the final dead load configuration. In the final state, the maximum unbalanced joint force, in any direction, does not exceed 0·1 kips at any joint.

TABLE 5. DATA FOR ASSUMED CONFIGURATION OF PROBLEM 3

Description		Magnitude
Cross-sectional area of cables		1·5 in^2
Support stiffness	Longitudinal	10,000,000 kips/ft
	Transverse	10,000,000 kips/ft
	Vertical	10,000,000 kips/ft
Young's modulus of elasticity		22,000 kips/in^2
Dead load	Interior joints	34·375 kips
	Exterior joints	17·188 kips
Horizontal component of prestressing force in all members		225·0 kips

TABLE 6. INITIAL GEOMETRY FOR PROBLEM 3

Joint	Joint coordinates		
	x (ft)	y (ft)	z (ft)
(1, 2)	0·0	5.555556	25·0
(1, 3)	0·0	8.888889	50·0
(1, 4)	0·0	10·000000	75·0
(2, 2)	25·0	0·000000	25·0
(2, 3)	25·0	3·333333	50·0
(2, 4)	25·0	4·444443	75·0
(3, 2)	50·0	−3·333333	25·0
(3, 3)	50·0	0·000000	50·0
(3, 4)	50·0	1·111110	75·0
(4, 2)	75·0	−4·444443	25·0
(4, 3)	75·0	−1·111110	50·0
(4, 4)	75·0	0·000000	75·0
(2, 1)	25·0	−5·555556	0·0
(3, 1)	50·0	−8·888889	0·0
(4, 1)	75·0	−10·000000	0·0

TABLE 7. EQUILIBRIUM CONFIGURATION UNDER DEAD LOAD FOR PROBLEM 3

Joint	Joint coordinates		
	x (ft)	y (ft)	z (ft)
(1, 2)	0·00001256	5·55560000	25·00000000
(1, 3)	0·00000839	8·88890000	50·00000000
(1, 4)	0·00000720	10·00000000	75·00000000
(2, 2)	25·12600000	1·03150000	24·86500000
(2, 3)	25·17700000	4·86170000	49·90200000
(2, 4)	25·19200000	6·12460000	75·00000000
(3, 2)	50·07200000	−2·13400000	24·84200000
(3, 3)	50·10300000	1·79370000	49·88400000
(3, 4)	50·11200000	3·08500000	75·00000000
(4, 2)	74·99900000	−3·22100000	24·83800000
(4, 3)	74·99900000	0·72151000	49·88100000
(4, 4)	74·99900000	2·01700000	75·00000000
(2, 1)	25·00000000	−5·55555555	0·00003647
(3, 1)	50·00000000	−8·88890000	0·00003901
(4, 1)	75·00000000	−10·00000000	0·00003938

TABLE 8. CABLE TENSIONS AT EQUILIBRIUM CONFIGURATION UNDER DEAD LOAD FOR PROBLEM 3

Cable element		Cable forces in kips
Longitudinal	$L2, 1$	127·63
	$L2, 2$	127·73
	$L2, 3$	127·49
	$L3, 1$	84·97
	$L3, 2$	85·09
	$L3, 3$	84·86
	$L4, 1$	72·85
	$L4, 2$	72·96
	$L4, 3$	72·75
Transverse	$T2, 1$	377·34
	$T2, 2$	368·43
	$T2, 3$	364·39
	$T3, 1$	404·23
	$T3, 2$	394·73
	$T3, 3$	390·43
	$T4, 1$	408·15
	$T4, 2$	398·56
	$T4, 3$	394·22

The total dead load on the structure is 1203·125 kips (55 psf) and the support reactions add up to 1201·097 kips acting vertically upward. Since there are no horizontal components of dead load, the total horizontal support reactions in both the longitudinal and transverse directions are zero.

This dead load equilibrium state would now serve as the initial configuration for the application of live load to the structure.

6. SUMMARY AND CONCLUSIONS

A method of analysis is developed for studying the response of single-layer cable nets. The full three-dimensional response of the cable system is considered and any general loading condition, including temperature change, is permissible.

The detailed development presented presumes a doubly threaded net of intersecting longitudinal and transverse cables; however, the general scheme could be adopted to more complicated meshes. The intersecting cables need not be orthogonal and the spacing need not be uniform. The cable supports may be rigid or flexible and hangers may be included at the node points.

The mathematical model employed reflects a discrete system of cable elements. The governing equations of equilibrium are nonlinear in the unknown displacement quantities. These displacements are determined by the Newton–Raphson method. Each linear cycle is a boundary-value problem which is solved as a set of initial-value problems. Computational sensitivities are successfully controlled by a "suppression" technique for each linear solution. No convergence problems were encountered in applying the Newton–Raphson method to a wide variety of problems.

The method of analysis is designed primarily to determine the forces and displacements that result from prescribed live loads and temperature changes. However, it can also be used to determine the initial dead load configuration. Example problems illustrate both applications.

The discrete model coupled with the initial-value solution technique provides a method for a very realistic analysis of cable nets without having to directly solve large numbers of nonlinear simultaneous algebraic equations. This technique thus has computational advantages unless a large number of suppressions is required.

Acknowledgements—The material in this paper was developed as part of a project supported by the National Science Foundation through a Research Initiation Grant which was awarded to the senior author. The authors acknowledge with thanks this support.

Thanks is also due Mr. Daniel Caramanico, who assisted in the early phases of the study program.

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APPENDIX

Equations of equilibrium

In this Appendix, the derivation of the basic equations of equilibrium for the cable system is outlined. Definitions of terms already introduced in the text are not repeated here; however, all new terms are defined.

Taking V_T as the total potential energy of the system, and ϕ_i as one of the k admissible displacements, we have the following k equations of equilibrium:

$$\frac{\partial V_T}{\partial \phi_i} = 0 \qquad (i = 1, 2, \ldots, k). \tag{7}$$

The detailed derivation of V_T is too long for inclusion here. It is, however, straightforward and is given in detail by Kar [10]. This is simply an extension to three dimensions of the two-dimensional case given by Caramanico [11] and West and Caramanico [8].

At a typical joint (ij) on the cable net, the differentiation indicated in equation (7) is performed with respect to u_{ij}, v_{ij} and w_{ij} to yield, respectively, the equations of longitudinal, vertical and transverse equilibrium at point (ij).

Thus, for the arrangement shown in Fig. 1, $\partial V_T/\partial u_{ij} = 0$ yields

$$\begin{aligned}
&-\alpha_{Ljh}u_{hj}-\alpha_{Tii}u_{ii}+(\alpha_{Ljh}+\alpha_{Lji}+\alpha_{Tii}+\alpha_{Tij}+K_{xij})u_{ij}-\alpha_{Tij}u_{ik}\\
&-\alpha_{Lji}u_{jj}-\beta_{Ljh}v_{hj}-\beta_{Tii}v_{ii}+(\beta_{Ljh}+\beta_{Lji}+\beta_{Tii}+\beta_{Tij})v_{ij}-\beta_{Tij}v_{ik}\\
&-\beta_{Lji}v_{jj}-\xi_{Ljh}w_{hj}-\xi_{Tii}w_{ii}+(\xi_{Ljh}+\xi_{Lji}+\xi_{Tii}+\xi_{Tij})w_{ij}-\xi_{Tij}w_{ik}\\
&-\xi_{Lji}w_{jj}+N_{xij} = -X_{Ljh}+X_{Lji}-X_{Tii}+X_{Tij}+W_{xij}+Q_{xij}+Q^t_{xij}.
\end{aligned} \tag{8}$$

Similarly, $\partial V_T/\partial v_{ij} = 0$ and $\partial V_T/\partial w_{ij} = 0$ produce, respectively

$$\begin{aligned}
&-\beta_{Ljh}u_{hj}-\beta_{Tii}u_{ii}+(\beta_{Ljh}+\beta_{Lji}+\beta_{Tii}+\beta_{Tij})u_{ij}-\beta_{Tij}u_{ik}-\beta_{Lji}u_{jj}\\
&-\gamma_{Ljh}v_{hj}-\gamma_{Tii}v_{ii}+(\gamma_{Ljh}+\gamma_{Lji}+\gamma_{Tii}+\gamma_{Tij}+K_{yij})v_{ij}-\gamma_{Tij}v_{ik}\\
&-\gamma_{Lji}v_{jj}-\Delta_{Ljh}w_{hj}-\Delta_{Tii}w_{ii}+(\Delta_{Ljh}+\Delta_{Lji}+\Delta_{Tii}+\Delta_{Tij})w_{ij}-\Delta_{Tij}w_{ik}\\
&-\Delta_{Lji}w_{jj}+N_{yij} = -Y_{Ljh}+Y_{Lji}-Y_{Tii}+Y_{Tij}+W_{yij}+Q_{yij}+Q^t_{yij}
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
&-\xi_{Ljh}u_{hj}-\xi_{Tii}u_{ii}+(\xi_{Ljh}+\xi_{Lji}+\xi_{Tii}+\xi_{Tij})u_{ij}-\xi_{Tij}u_{ik}-\xi_{Lji}u_{jj}\\
&-\Delta_{Ljh}v_{hj}-\Delta_{Tii}v_{ii}+(\Delta_{Ljh}+\Delta_{Lji}+\Delta_{Tii}+\Delta_{Tij})v_{ij}-\Delta_{Tij}v_{ik}-\Delta_{Lji}v_{jj}\\
&-\varepsilon_{Ljh}w_{hj}-\varepsilon_{Tii}w_{ii}+(\varepsilon_{Ljh}+\varepsilon_{Lji}+\varepsilon_{Tii}+\varepsilon_{Tij}+K_{zij})w_{ij}-\varepsilon_{Tij}w_{ik}\\
&-\varepsilon_{Lji}w_{jj}+N_{zij} = -Z_{Ljh}+Z_{Lji}-Z_{Tii}+Z_{Tij}+W_{zij}+Q_{zij}+Q^t_{zij}
\end{aligned} \tag{10}$$

where

$$\alpha_l = \left[K\left(\frac{h_d^2}{L_d^2}+\frac{(e_d-e_t)}{L_d}-\frac{h_d^2(e_d-e_t)}{L_d^3}\right)\right]_l$$

$$\beta_l = \left[K\left(\frac{h_d r_d}{L_d^2}-\frac{h_d r_d(e_d-e_t)}{L_d^3}\right)\right]_l$$

$$\gamma_l = \left[K\left(\frac{r_d^2}{L_d^2}+\frac{(e_d-e_t)}{L_d}-\frac{r_d^2(e_d-e_t)}{L_d^3}\right)\right]_l$$

$$\xi_l = \left[K\left(\frac{h_d s_d}{L_d^2}-\frac{h_d s_d(e_d-e_t)}{L_d^3}\right)\right]_l$$

$$\Delta_l = \left[K\left(\frac{r_d s_d}{L_d^2}-\frac{r_d s_d(e_d-e_t)}{L_d^3}\right)\right]_l$$

$$\varepsilon_l = \left[K\left(\frac{s_d^2}{L_d^2}+\frac{(e_d-e_t)}{L_d}-\frac{s_d^2(e_d-e_t)}{L_d^3}\right)\right]_l$$

$$X_l = \left[\frac{K e_d h_d}{L_d}\right]_l$$

$$Y_l = \left[\frac{Ke_d r_d}{L_d}\right]_l$$

$$Z_l = \left[\frac{Ke_d s_d}{L_d}\right]_l$$

$$N_{xij} = -\sum_l \left\{ KL_d[L_d-(e_d-e_t)]\frac{\partial R_l}{\partial u_{ij}} \right\}_l$$

$$N_{yij} = -\sum_l \left\{ KL_d[L_d-(e_d-e_t)]\frac{\partial R_l}{\partial v_{ij}} \right\}_l$$

$$N_{zij} = -\sum_l \left\{ KL_d[L_d-(e_d-e_t)]\frac{\partial R_l}{\partial w_{ij}} \right\}_l$$

$$R_l = \sqrt{(1+\Phi_l)} - \left\{ 1 + \frac{(\bar{u}_m-u_i)^2}{2L_d^2} + \frac{(\bar{v}_m-v_i)^2}{2L_d^2} + \frac{(\bar{w}_m-w_i)^2}{2L_d^2} \right.$$

$$+ \frac{h_d(\bar{u}_m-u_i)}{L_d^2} + \frac{r_d(\bar{v}_m-v_m)}{L_d^2} + \frac{s_d(\bar{w}_m-w_i)}{L_d^2} - \frac{h_d^2(\bar{u}_m-u_i)^2}{2L_d^4}$$

$$- \frac{r_d^2(\bar{v}_m-\bar{v}_i)^2}{2L_d^4} - \frac{s_d^2(\bar{w}_m-w_i)^2}{2L_d^4} - h_d r_d(\bar{u}_m-u_i)(\bar{v}_m-v_i)$$

$$\left. - \frac{h_d s_d(\bar{u}_m-u_i)(\bar{w}_m-w_i)}{L_d^4} - \frac{r_d s_d(\bar{u}_m-u_i)(\bar{w}_m-w_i)}{L_d^4} \right\}_l$$

$$\Phi_l = \left\{ \frac{(\bar{u}_m-u_i)^2}{L_d^2} + \frac{(\bar{v}_m-v_i)^2}{L_d^2} + \frac{(\bar{w}_m-w_i)^2}{L_d^2} + \frac{2h_d(\bar{u}_m-u_i)}{L_d^2} \right.$$

$$\left. + \frac{2r_d(\bar{v}_m-v_i)}{L_d^2} + \frac{2s_d(\bar{w}_m-w_i)}{L_d^2} \right\}_l .$$

The subscript l in the above terms is associated with the members which frame into joint (ij) and ranges over elements Ljh, Lji, Tii and Tij as previously shown in Fig. 1. All quantities within a bracket carrying the l subscript corresponds to the lth element. These terms are defined as follows:

h_d, r_d, s_d = longitudinal, vertical and transverse projections of cable element at initial configuration

L_d = length of cable element at initial configuration

e_d = elongation of cable element at initial configuration

e_t = elongation of cable element associated with temperature change

E = modulus of elasticity

A = cross-sectional area of cable element

L_o = unstressed length of cable element

$K = \frac{EA}{L_o}$ = axial stiffness of cable element

$\bar{u}_m, \bar{v}_m, \bar{w}_m$ = longitudinal, vertical and transverse displacements, respectively, at the distant end of member l from point (ij).

Additional terms in equations (8, 9 and 10) which are joint oriented are:

$W_{xij}, W_{yij}, W_{zij}$ = longitudinal, vertical and transverse components of dead load at (ij)

$Q_{xij}, Q_{yij}, Q_{zij}$ = longitudinal, vertical and transverse components of applied load at (ij)

$$Q^t_{xij} = -\left[\frac{Ke_t h_d}{L_d}\right]_{Lji} + \left[\frac{Ke_t h_d}{L_d}\right]_{Ljh} - \left[\frac{Ke_t h_d}{L_d}\right]_{Tij} + \left[\frac{Ke_t h_d}{L_d}\right]_{Tii}$$

$$Q^t_{yij} = -\left[\frac{Ke_t r_d}{L_d}\right]_{Lji} + \left[\frac{Ke_t r_d}{L_d}\right]_{Ljh} - \left[\frac{Ke_t r_d}{L_d}\right]_{Tij} + \left[\frac{Ke_t r_d}{L_d}\right]_{Tii}$$

$$Q^t_{zij} = -\left[\frac{Ke_t s_d}{L_d}\right]_{Lji} + \left[\frac{Ke_t s_d}{L_d}\right]_{Ljh} - \left[\frac{Ke_t s_d}{L_d}\right]_{Tij} + \left[\frac{Ke_t s_d}{L_d}\right]_{Tii}.$$

The last three terms are the equivalent temperature loads in the longitudinal, vertical and transverse directions, respectively, at point (ij). The subscript on the brackets indicates the member associated with all the quantities within the bracket.

The quantity X_l represents the longitudinal force induced at point (ij) from the initial configuration load in the lth bar. Thus, if the structure is in equilibrium at this initial configuration,

$$-X_{Ljh} + X_{Lji} - X_{Tii} + X_{Tij} + W_{xij} = 0 \tag{11}$$

and the right-hand side of equation (8) reduces to $Q_{xij} + Q^t_{xij}$. Similarly, the right-hand sides of equations (9 and 10) simplify for a structure initially in equilibrium. The only loads remaining after simplification are the live loads and temperature loads.

(*Received* 3 *August* 1972)

Абстракт—На основе дискретной математической модели, дается метод анализа однослойных канатных сеток. Определяющие уравнения нелинейны, выражены в перемещениях. Они решаются методом Ньютона-Рафсона. Каждый линейный цикл является краевой задачей, решеной в виде системы задач на начальные значения. Метод анализа первостепенно касается определения усилий и перемещений, происходящих в результате переменной нагрузки и изменений температуры. Тем не менее, пользуясь этим методом можно, также, определить начальную конфигурацию собственной нагрузки. Примеры иллюстрируют оба применения.